ASYMPTOTIC LAWS OF BEHAVIOR OF DETONATION WAVES

PMM Vol. 31, No. 3, 1967, pp. 393-405

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(*Received January 25, 1967*)

The asymptotic laws of shock waves propagation in a quiescent homogenous gas depend in general on conditions which define the motions of gas in the disturbed region behind the shock wave, and may vary considerably. These laws were the subject of detailed analysis in a number of works which dealt with plane, cylindrical and spherical shock waves under conditions in which gas motions behind the wave weakened the shock wave with consequent degeneration of the latter into an acoustic wave. For plane, cylindrical and spherical wave propagation these asymptotic laws of propagation are formulated as follows [1]

$$a_1(t-t_0)=r_s\left[1-\left(\frac{r_0}{r_s}\right)^{1/2}-\frac{1}{8}\frac{r_0}{r_s}\ln\frac{r_s}{r_0}+\ldots\right]$$
$$a_1(t-t_0)=r_s\left[1-2\left(\frac{r_0}{r_s}\right)^{3/4}+\ldots\right] \qquad (1)$$
$$a_1(t-t_0)=r_s\left[1-C\frac{r_0}{r_s}\left(\ln\frac{r_s}{r_0}\right)^{1/2}+\ldots\right]$$

Here a_1 is the velocity of sound in the quiescent gas, t is time, r_s the shock wave coordinate, and t_0 and r_0 are certain constants.

The degeneration of a shock wave into an acoustic one takes place in accordance with the asymptotic Formulas (1) at infinity only, and the shock wave has no asymptote in the rt-plane, receding to any distance away from an arbitrary straight line $r-a_1(t-t_0) =$ $=$ const.

An analysis is made in this paper of the asymptotic laws of detonation wave propagation under conditions in which a strong detonation wave is weakened by gas motions behind it, and transformed into a Chapman – Jouguet wave. It is shown that in contrast to the asymptotic behavior of shock waves, a strong plane detonation wave tends at infinity to the asymptote $r-c_J(t-t_0)=$ const (c_J is the propagation velocity of the Chapman–Jouguet detonation wave), while the transformation of strong cylindrical, or spherical detonation waves into a Chapman – Jouguet wave may, in general, occur at finite distances. The flow pattern of cylindrical and spherical waves after these have reached the Chapman – Jouguet mode is also studied.

A brief account of the results of this work on plane waves is given in [2].

Let v, p, and ρ be respectively the gas velocity, pressure, and density, c the detonation wave propagation velocity, and γ the gas specific heat ratio; subscript 1 denotes the pressure and density of the gas at rest. The conditions at the detonation wave may then be written as follows:

$$-\rho_1c=\rho(v-c),\qquad \rho_1c^2+p_1=\rho(v-c)^2+p$$
$$\frac{1}{2}c^2+\frac{a_1^2}{\gamma-1}+Q=\frac{1}{2}(v-c)^2+\frac{\gamma}{\gamma-1}\frac{p}{\rho} \qquad (2)$$

Here Q is the heat release in a unit mass of gas. Solving Eqs. (2) for v, p, and ρ, we obtain

$$v=\frac{a_1}{\gamma+1}\,\frac{1-q+\sqrt{(1-qq_J)(1-q/q_J)}}{\sqrt{q}}$$

$$p=p_1+\frac{\rho_1 a_1 v}{\sqrt{q}},\qquad \rho=\frac{\rho_1}{1-v\sqrt{q}/a_1} \tag{3}$$

Here $q = a_1{}^2/c^2$, and q_J is the value of q corresponding to the Chapman – Jouguet detonation wave velocity, defined by

$$\frac{a_1}{\gamma+1}\,\frac{1-q_J}{\sqrt{q_J}}=\left(2\frac{\gamma-1}{\gamma+1}Q\right)^{1/2}$$

Let us consider expressions defining the gas parameters behind a detonation wave of an intensity only slightly higher than that of the Chapman – Jouguet wave. For the definition of the deviation of the detonation wave from that of the Chapman – Jouguet wave we introduce parameter $\varepsilon = 1 - q/q_J$. With the aid of this parameter we shall present Expressions (3) as follows:

$$v=\frac{c_J}{\gamma+1}(1-\varepsilon)^{-1/2}\left(1-q_J+q_J\varepsilon+\varepsilon^{1/2}\sqrt{1-q_J{}^2+\varepsilon q_J{}^2}\right)$$

$$\frac{p}{p_1}=1+\frac{\gamma c_J v}{a_1{}^2}(1-\varepsilon)^{-1/2},\qquad \frac{\rho_1}{\rho}=1-\frac{v}{c_J}(1-\varepsilon)^{1/2} \tag{4}$$

On the assumption that parameter ε is small we expand the right-hand sides of Eqs. (4) into power series of this parameter, and limit our expansions to terms containing its first power (we assume the detonation wave to be sufficiently strong, so that q_J is not close to unity). We then obtain

$$\frac{v}{v_J}=1+\left(\frac{1+q_J}{1-q_J}\right)^{1/2}\varepsilon^{1/2}+\frac{1+q_J}{2(1-q_J)}\varepsilon+\cdots$$

$$\frac{p}{p_J}=1+\gamma\frac{1-q_J}{\gamma+q_J}\left[\left(\frac{1+q_J}{1-q_J}\right)^{1/2}\varepsilon^{1/2}+\frac{1}{1-q_J}\varepsilon+\cdots\right] \tag{5}$$

$$\frac{\rho_J}{\rho}=1-\frac{1-q_J}{\gamma+q_J}\left[\left(\frac{1+q_J}{1-q_J}\right)^{1/2}\varepsilon^{1/2}+\frac{1}{1-q_J}\varepsilon+\cdots\right]$$

Here v_J, p_J, ρ_J are respectively the values of velocity, pressure and density of the gas behind the Chapman – Jouguet wave defined as follows:

$$v_J=\frac{c_J}{\gamma+1}(1-q_J)=\frac{a_J}{\gamma+q_J}(1-q_J),\qquad p_J=p_1+\rho_1 c_J v_J=\rho_1 c_J{}^2\frac{\gamma+q_J}{\gamma(\gamma+1)}$$

$$\rho_J=\rho_1\frac{c_J}{c_J-v_J}=\rho_1\frac{c_J}{a_J}=\rho_1\frac{\gamma+1}{\gamma+q_J}$$

From (5) we easily derive

$$\frac{p}{\rho^\gamma}=\frac{p_J}{\rho_J{}^\gamma}\left[1+\frac{\gamma(\gamma-1)(1-q_J)^2}{2(\gamma+q_J)^2}\varepsilon+O(\varepsilon^{3/2})\right]$$

$$a-\frac{\gamma-1}{2}v=a_J-\frac{\gamma-1}{2}v_J+\frac{1}{4}v_J\left[\gamma-1-\frac{(\gamma+1)^2}{2}\frac{1+q_J}{\gamma+q_J}\right]\varepsilon+O(\varepsilon^{3/2})$$

It follows from this that the gas parameters behind detonation waves close to the Chapman – Jouguet waves satisfy the same relationships as the Riemann travelling waves, with an approximation of the order up to and including $\varepsilon^{1/2}$. We shall make use of this deduction later, when considering the asymptotic behavior of plane detonation waves. We note that for ordinary shock waves ($q_J = 1$) parameters p/ρ^γ and $a - \frac{1}{2}(\gamma - 1)v$ remain constant behind the wave, if terms up to and including ε^2 are considered.

The equations of one-dimensional motions of gas

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial r} + (\nu - 1)\frac{\rho v}{r} = 0, \qquad \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0$$

$$\frac{\partial}{\partial t}\frac{p}{\rho^\gamma} + v\frac{\partial}{\partial r}\frac{p}{\rho^\gamma} = 0 \tag{6}$$

(here $\nu = 1, 2, 3$ corresponds respectively to motions in the presence of plane, cylindrical, or spherical waves), together with conditions (2) at the detonation wave make it possible to express the derivatives of gasdynamic parameters with respect to coordinate r behind the wave by means of gas parameters of the latter, as well as parameter dq/dr which defines the wave acceleration. We, thus, obtain the expression for derivative $\partial v/\partial r|_s$

$$\left.\frac{\partial v}{\partial r}\right|_s = \frac{2(1+q) + \sqrt{(1-qq_J)(1-q/q_J)}}{2(1-qq_J)(1-q/q_J)}\frac{v}{q}\frac{dq}{dr} - \frac{(\nu-1)qv}{r}\frac{1+\gamma v/a_1\sqrt{q}}{\sqrt{(1-qq_J)(1-q/q_J)}} \tag{7}$$

We shall begin by considering a plane detonation wave ($\nu = 1$). In this case the flow behind a detonation wave represents, as was shown above, a Riemann travelling wave with an approximation up to and including terms of the order of $\varepsilon^{1/2}$. For such a wave we have

$$v = \Phi\,[r - (a + v)\,t], \quad a - {}^1/_2(\gamma - 1)\,v = a_J - {}^1/_2(\gamma - 1)v_J \tag{8}$$

where a is the velocity of sound, and Φ an arbitrary function the form of which determines the type of the travelling wave. Let us assume that function $\Phi(\xi)$ is such that $\Phi(\xi_0) = v_J$, and that $r\Phi'(\xi) \to \infty$ when $r \to \infty$ and $\xi \to \xi_0$, with ξ_0 being the limit of $\xi = r - (a + v)\,t$ at the detonation wave.

From Eqs. (8) we easily deduce that

$$\frac{\partial v}{\partial r} = \frac{\Phi'(\xi)}{1 + {}^1/_2(\gamma+1)\,t\Phi'(\xi)}$$

With $r \to \infty$ a detonation wave tends to the Chapman – Jouguet detonation mode, so that $c_J t/r_s \to 1$. Consequently.

$$\left.\frac{\partial v}{\partial r}\right|_s \to \frac{2}{\gamma+1}\frac{c_J}{r_s} \quad \text{при} \quad r \to \infty$$

Substituting this expression for $\partial v/\partial r|_s$ in Formula (7), and retaining in its right-hand side the main terms of ε only, we derive the equation which leads to the establishment of the asymptotic law of detonation wave propagation

$$\frac{2}{r_s} = -\frac{1}{\varepsilon}\frac{d\varepsilon}{dr_s}$$

Integration of this yields

$$\varepsilon r_s^2 = r_0^2 \;\; \text{или} \;\; \left[1 - c_J\left(\frac{dt}{dr_s}\right)^2\right] r_s^2 = r_0^2 \qquad (r_0 = \text{const})$$

Integrating once more, and using Expressions (5), we find the asymptotic law of plain detonation wave propagation, as well as asymptotic formulas for parameters of the gas behind the wave

$$c_J(t - t_0) = r_s\left(1 + \frac{r_0^2}{2r_s^2} + \cdots\right)$$

$$\frac{v}{v_J} = 1 + \left(\frac{1+q_J}{1-q_J}\right)^{1/2}\frac{r_0}{r_s} + \cdots$$

$$\frac{p}{p_J} = 1 + \gamma\frac{1-q_J}{\gamma+q_J}\left(\frac{1+q_J}{1-q_J}\right)^{1/2}\frac{r_0}{r_s} + \cdots$$

$$\frac{\rho}{\rho_J} = 1 + \frac{1-q_J}{\gamma+q_J}\left(\frac{1+q_J}{1-q_J}\right)^{1/2}\frac{r_0}{r_s} + \cdots \tag{9}$$

(r_0 is a certain constant)

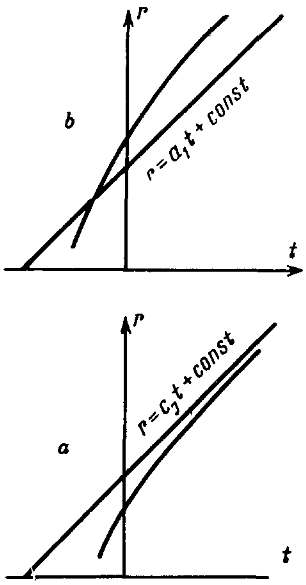

Fig. 1

It follows from Formulas (9) that a plane detonation wave degenerating into a Chapman – Jouguet wave tends towards asymptote

$$r - c_J t = \text{const}$$

This behavior differs substantially from that of the asymptotic behavior of an ordinary plane shock wave degenerating into an acoustic wave. According to the first of Eqs. (1) a shock wave has no asymptote, and intersects the straight line $r - a_1 t$ = const for any large value of the constant in the straight line equation. The difference between the asymptotic behavior of a plane detonation wave (curve a) and a plane shock wave (curve b) is shown on Fig. 1.

We shall now show that transition to the Chapman – Jouguet detonation pattern of flows in the presence of cylindrical, or spherical detonation waves propagating in a gas at rest differs from flows in the presence of plane waves that it may occur at a finite distance.

We revert to Eq. (7) which defines the derivative $\partial v/\partial r$ at points of a detonation wave. For small ε this becomes

$$r_s \frac{\partial v}{\partial r}\bigg|_s = -\frac{v_J}{1-q_J}\frac{r_s}{\varepsilon}\frac{d\varepsilon}{dr_s} - (\nu - 1)\frac{1+\gamma v_J/a_1\sqrt{q_J}}{\sqrt{1-q_J^2}\,\varepsilon^{1/2}} \qquad (10)$$

It follows from this that for self-similar motions with $\nu \neq 1$, the negative magnitude $r_s \partial v/\partial r|_s$ tends in its absolute value to infinity at the rate of $\varepsilon^{-1/2}$, when ε tends to zero Assuming further that the flow behind the detonation wave weakens the latter so that with ε decreasing to zero the absolute value of $r_s \partial v/\partial r|_s$ tends to infinity at a slower rate than in the case of self-similar motions. We now obtain from Eq. (10) the following asymptotic Eq.:

$$r_s \frac{d\varepsilon}{dr_s} = -N\varepsilon^{1/2} \qquad (N > 0)$$

Integration of this equation yields

$$\varepsilon^{1/2} = \varepsilon_0^{1/2} - \frac{N}{2}\ln\frac{r_s}{r_{s0}}$$

It will be seen from this formula that ε becomes zero at a finite value of r_s, therefore the transition to the Chapman – Jouguet detonation pattern occurs at a finite distance.

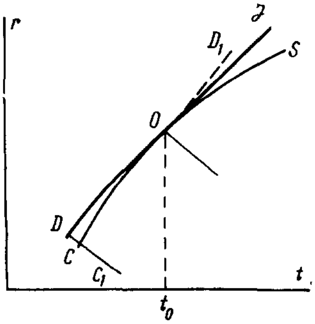

Fig. 2

We shall study the conditions under which the above assumption is fulfilled, and shall determine the flow in the neighborhood of the point of transition to the Chapman – Jouguet mode.

Let detonation wave DO (Fig. 2) be gradually weakened so that parameter ε becomes zero at point O, at $t = t_0$, and then remains constant with further increases of t. We select the origin of time so that the equation of the Chapman – Jouguet OJ wave may be written in the form of $r = c_J t$.

We substitute in the Eqs. of motion (6) for the unknown functions (v, p, ρ) and the independent parameters r, t the following new variables

$$v = v_J V, \quad p = p_J P, \quad \rho = \rho_J R, \quad \lambda = \frac{r}{c_J t}, \quad \tau = \ln\frac{t}{t_0}$$

After transformation, we obtain

$$\frac{\partial R}{\partial \tau}+\left(\frac{1-q}{\gamma+1}V-\lambda\right)\frac{\partial R}{\partial \lambda}+\frac{1-q}{\gamma+1}R\frac{\partial V}{\partial \lambda}+\frac{(\nu-1)(1-q)}{\gamma+1}\frac{RV}{\lambda}=0$$

$$R\left[\frac{\partial V}{\partial \tau}+\left(\frac{1-q}{\gamma+1}V-\lambda\right)\frac{\partial V}{\partial \lambda}\right]+\frac{(\gamma+q)^2}{\gamma(\gamma+1)(1-q)}\frac{\partial P}{\partial \lambda}=0 \tag{11}$$

$$\frac{\partial P}{\partial \tau}-\gamma\frac{P}{R}\frac{\partial R}{\partial \tau}+\left(\frac{1-q}{\gamma+1}V-\lambda\right)\left(\frac{\partial P}{\partial \lambda}-\gamma\frac{P}{R}\frac{\partial R}{\partial \lambda}\right)=0$$

We shall seek the solution of system (11) in the neighborhood of point O behind the detonation wave. It is easy to establish that system (11) has three sets of characteristics, and that along the characteristics of the first two sets (acoustic) the following relationships must be fulfilled.

$$\lambda^{\cdot}+\lambda-\frac{1-q}{\gamma+1}V=\pm\frac{\gamma+q}{\gamma+1}\left(\frac{P}{R}\right)^{1/2} \tag{12a}$$

$$\frac{1}{\gamma(1-q)}\frac{P^{\cdot}}{P}+\frac{\gamma+1}{(\gamma+q)^2}\frac{RV^{\cdot}}{P}\left(\lambda^{\cdot}+\lambda-\frac{1-q}{\gamma+1}V\right)+\frac{\nu-1}{\gamma+1}\frac{V}{\lambda}=0 \tag{12b}$$

and along the characteristic of the third set (particle trajectories)

$$\lambda^{\cdot}+\lambda-\frac{1-q}{\gamma+1}V=0, \qquad RP^{\cdot}-\gamma PR^{\cdot}=0 \tag{12c}$$

Here, the dot denotes differentiation along the characteristic with respect to τ.

If the values of functions $V=V_0(\tau)$, $P=P_0(\tau)$, $R=R_0(\tau)$ do not satisfy the characteristic relationships along a certain line $\lambda=\lambda_0(\tau)$, then the solution of Eqs. (11) in the neighborhood of this line may be expressed by expansion

$$V-V_0=V_1^*(\lambda-\lambda_0)+V_2^*(\lambda-\lambda_0)^2+\ldots \tag{13}$$

with similar expansions for P and R. Consecutive coefficients of these expansions are uniquely defined by the values of functions along line $\lambda=\lambda_0(\tau)$. When the initial data fulfil the characteristic relationships, then one of the first coefficients of series (13) may, for a certain value of τ, be selected arbitrarily (for example $V_1^*(0)$ for solutions with initial data on the characteristic of the first, or second set, and $R_1^*(0)$ with initial data on the third set characteristic).

Let functions λ_0, V_0, P_0 and R_0 conform to one of the first characteristic relationships (12a), but not to the second. This means that line $\lambda=\lambda_0(\tau)$ is an envelope of the characteristics.

In this case there are no solutions in the form of (13). There exist, however, solutions of Eqs. (11) in the form of series as follows:

$$V=V_0+V_1\sqrt{\lambda_0-\lambda}+V_2(\lambda_0-\lambda)+\cdots$$

$$P=P_0+P_1\sqrt{\lambda_0-\lambda}+P_2(\lambda_0-\lambda)+\cdots \tag{14}$$

$$R=R_0+R_1\sqrt{\lambda_0-\lambda}+R_2(\lambda_0-\lambda)+\cdots$$

the coefficients of which are uniquely defined by initial data along line $\lambda=\lambda_0(\tau)$. In fact, a substitution of expansions (14) into Eqs. (11) yields a system of relationships which permits a consecutive determination of coefficients of these. We write down the first two of such systems

$$\frac{1-q}{\gamma+1}R_0V_1-\left(\lambda_0^{\cdot}+\lambda_0-\frac{1-q}{\gamma+1}V_0\right)R_1=0$$

$$R_0\left(\lambda_0^{\cdot}+\lambda_0-\frac{1-q}{\gamma+1}V_0\right)V_1-\frac{(\gamma+q)^2}{\gamma(\gamma+1)(1-q)}P_1=0 \tag{15a}$$

$$\left(\lambda_0^{\cdot}+\lambda_0-\frac{1-q}{\gamma+1}V_0\right)(R_0P_1-\gamma P_0R_1)=0$$

$$\frac{1-q}{\gamma+1} R_0 V_2 - \left(\lambda_0^{\cdot} + \lambda_0 - \frac{1-q}{\gamma+1} V_0\right) R_2 = R_0^{\cdot} + \frac{(\nu-1)(1-q)}{\gamma+1} \frac{R_0 V_0}{\lambda_0} - \frac{1-q}{\gamma+1} R_1 V_1$$

$$2R_0\left(\lambda_0^{\cdot} + \lambda_0 - \frac{1-q}{\gamma+1} V_0\right) V_2 - \frac{(\gamma+q)^2}{\gamma(\gamma+1)(1-q)} P_2 = -2R_0 V_0^{\cdot} + $$
$$+ \frac{1-q}{\gamma+1} R_0 V_1^2 - \left(\lambda_0^{\cdot} + \lambda_0 - \frac{1-q}{\gamma+1} V_0\right) R_1 V_1 \qquad (15b)$$

$$2\left(\lambda_0^{\cdot} + \lambda_0 - \frac{1-q}{\gamma+1} V_0\right)(R_0 P_2 - \gamma P_0 R_2) = -2(R_0 P_0^{\cdot} - \gamma P_0 R_0^{\cdot}) +$$
$$+ (\gamma-1)\left(\lambda_0^{\cdot} + \lambda_0 - \frac{1-q}{\gamma+1} V_0\right) P_1 R_1 + \frac{1-q}{\gamma+1}(R_0 P_1 - \gamma P_0 R_1) V_1$$

In consequence of the assumption that functions λ_0, V_0, P_0, and R_0 satisfy one of the conditions (12a), the determinant of coefficients of V_1, P_1, R_1 in the first system of relationships is *equal to zero*. Then

$$P_1 = \gamma \frac{1-q}{\gamma+1} \frac{P_0 V_1}{\lambda_0^{\cdot} + \lambda_0 - V_0(1-q)/(\gamma+1)}$$
$$R_1 = \frac{1-q}{\gamma+1} \frac{R_0 V_1}{\lambda_0^{\cdot} + \lambda_0 - V_0(1-q)/(\gamma+1)} \qquad (16)$$

As the determinant of system (15b) with respect to V_2, P_2, R_2 coincides with that of system (15a), and is, therefore, also zero, the determination of parameters V_2, P_2, R_2 requires the fulfilment of the known condition, which together with Eqs. (16) makes it possible to derive the following expression for V_1

$$V_1^2 = 2\frac{\gamma+1}{1-q}\left(\lambda_0^{\cdot} + \lambda_0 - \frac{1-q}{\gamma+1} V_0\right)\left[\frac{1}{\gamma(1-q)} \frac{P_0^{\cdot}}{P_0} + \right. \qquad (17)$$
$$\left. + \frac{\gamma+1}{(\gamma+q)^2} \frac{R_0 V_0^{\cdot}}{P_0}\left(\lambda_0^{\cdot} + \lambda_0 - \frac{1-q}{\gamma+1} V_0\right) + \frac{\nu-1}{\gamma+1} \frac{V_0}{\lambda_0}\right]$$

With this condition fulfilled, P_2 and R_2 may be defined in terms of V_2. For the determination of V_2 use is made of the solvability condition of the system for subsequent coefficients of expansions (14), etc.

Thus, expansions (14) yield the solution of Eqs. (11) which depends on the arbitrary functions λ_0, V_0, P_0, R_0 related to each other by one of Eqs. (12a). We note that when in Formula (17) we have $V_1 = 0$, then functions λ_0, V_0, P_0, R_0 also fulfil Eq. (12b), which means that the solution is determined by expansion (13) with an arbitrary value $V_1^*(0)$.

We shall use this solution for the construction of flows with strong cylindrical and spherical detonation waves which degenerate at finite distances into Chapman – Jouguet waves.

We shall begin by considering the flow behind a Chapman – Jouguet wave.

On the Chapman – Jouguet wave, i.e. with $\lambda = 1$, we obviously have $V = P = R = 1$. It is easy to prove that these initial data are not characteristic when $\nu \neq 1$, but satisfy the relationship (12a) with the upper sign. Therefore, the Chapman – Jouguet wave is the envelope of acoustic characteristics when $\nu \neq 1$. In accordance with previously made statements, the solution behind the wave will be of the form

$$V = 1 + V_1\sqrt{1-\lambda} + \cdots, \quad P = 1 + P_1\sqrt{1-\lambda} + \cdots$$
$$R = 1 + R_1\sqrt{1-\lambda} + \cdots$$

From Eqs. (16) and (17) we obtain

$$P_1 = \gamma R_1 = \gamma\frac{1-q}{\gamma+q} V_1, \qquad V_1 = \pm V_1^{\circ}, \qquad V_1^{\circ} = \left(2\frac{(\nu-1)(\gamma+q)}{(1-q)(\gamma+1)}\right)^{1/2}$$

All of the subsequent coefficients of series (14) are, clearly, constants as well, so that the flow behind the Chapman – Jouguet wave is necessarily self-similar, as though the detonation wave were a Chapman – Jouguet wave everywhere beginning at the instant $t = 0$. Depending on the selection of the sign of the expression of V_1, two different self-similar flows with a Chapman – Jouguet detonation wave are possible. With a positive sign we have a compression flow behind the detonation wave. Such flows occur in the presence of a cylindrical, or spherical piston expanding at a corresponding constant velocity. When the sign of the expression of V_1 is negative, we obtain a rarefaction flow. This flow may either extend continuously to the center $r = 0$, thus corresponding to the well known case of detonation wave propagation from a point (or line) ignition source, or it may join the compression flow via the compression shock, and extend up to the surface of the cylindrical, or spherical piston expanding at a constant velocity lower than that at which a compression flow is obtained throughout the region between the detonation wave and the piston.

The described self-similar flows are completely analogous to the self-similar stationary flows behind conical detonation waves analyzed in detail in works [3 and 4].

We shall now consider a flow behind that part of a detonation wave DO which precedes the onset of the Chapman – Jouguet mode. (Fig. 2).

We shall seek a solution in this region in the form of series (14), in which functions λ_0, V_0, P_0 and R_0 fulfil condition (12a) taken with the upper sign, i.e. condition

$$\lambda_0^{\cdot} + \lambda_0 - \frac{1-q}{\gamma+1} V_0 = \frac{\gamma+q}{\gamma+1} \left(\frac{P_0}{R_0}\right)^{1/2} \tag{18}$$

We shall limit our analysis to the small neighborhood of point O, and assume that for small τ functions λ_0, V_0, P_0 and R_0, and consequently all subsequent coefficients of series (14) may be represented by integral powers of τ, such as

$$\begin{aligned} V_0 &= 1 + V_{01}\tau + \ldots, & V_1 &= V_{10} + V_{11}\tau + \ldots, \\ P_0 &= 1 + P_{01}\tau + \ldots, & P_1 &= P_{10} + P_{11}\tau + \ldots \\ R_0 &= 1 + R_{01}\tau + \ldots, & R_1 &= R_{10} + R_{11}\tau + \ldots \\ \lambda_0 &= 1 + \lambda_1\tau + \lambda_2\tau^2 + \ldots \end{aligned} \tag{19}$$

From condition (18) we derive

$$\lambda_1 = 0, \qquad P_{01} - R_{01} + 2\frac{1-q}{\gamma+q} V_{01} - 4\frac{\gamma+1}{\gamma+q}\lambda_2 = 0 \tag{20}$$

From conditions (16) and (17) we find

$$P_{10} = \gamma\frac{1-q}{\gamma+q} V_{10}, \qquad R_{10} = \frac{1-q}{\gamma+q} V_{10} \tag{21}$$

$$V_{10}^2 = 2\frac{\gamma+q}{1-q}\left[\frac{P_{01}}{\gamma(1-q)} + \frac{V_{01}}{\gamma+q} + \frac{\nu-1}{\gamma+1}\right] \tag{22}$$

We rewrite Equations (5) at the detonation wave in the form

$$\lambda_D = 1 + d_2\tau^2 + d_3\tau^3 + \ldots$$

By virtue of

$$\varepsilon = 1 - (\lambda_D^{\cdot} + \lambda_D)^2$$

it follows from (5) that $d_2 = 0$, and

$$\begin{aligned} V_{01} - V_{10}\sqrt{\lambda_2} &= -\left(6\,\frac{1+q}{1-q} d_3\right)^{1/2} \\ P_{01} - P_{10}\sqrt{\lambda_2} &= -\gamma\frac{1-q}{\gamma+q}\left(6\,\frac{1+q}{1-q} d_3\right)^{1/2} \\ R_{01} - R_{10}\sqrt{\lambda_2} &= -\frac{1-q}{\gamma+q}\left(6\,\frac{1+q}{1-q} d_3\right)^{1/2} \end{aligned} \tag{23}$$

Formulas (20) to (23) define the seven parameters λ_2, V_{01}, P_{01}, R_{01}, V_{10}, P_{10} and R_{10}

in terms of d_3. Subsequent terms of these series may be similarly found for λ_0, V_0, ..., V_1, ...

It is convenient to express all magnitudes in Formulas (20) to (23) in terms of V_{10} as follows

$$P_{01} = \gamma R_{01} = \gamma \frac{1-q}{\gamma+q} V_{01} = \frac{4\gamma}{\gamma+q}\lambda_2 = \frac{\gamma(1-q)^2}{4(\gamma+q)}(V_{10}^2 - V_1^{\circ 2})$$

$$P_{10} = \gamma R_{10} = \gamma \frac{1-q}{\gamma+q} V_{10} \qquad (24)$$

$$V_{10}^2 - V_1^{\circ 2} - V_{10}\sqrt{V_{10}^2 - V_1^{\circ 2}} = -\frac{4}{1-q}\left(6\,\frac{1+q}{1-q}\,d_3\right)^{1/2} \equiv -d$$

The latter dependence, represented on Fig. 3, defines the relation between V_{10} and parameter d_3 which characterizes the wave form.

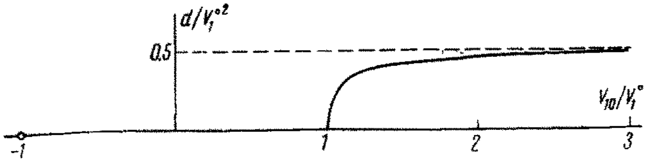

Fig. 3

For $d_3 = 0$ the solution is a two-valued one: $V_{10} = \pm V_1^{\circ}$. This corresponds to cases considered above in which either self-similar compression, or rarefaction flows occur behind the Chapman – Jouguet detonation wave. The solution is single-valued for all $d_3 > 0$, with parameter V_{10} increasing monotonously from V_1° with the increase of d_3, and tending to infinity when $d \to \frac{1}{2} V_1^{\circ 2}$. The solution in the region DCO (Fig. 2) bounded by the detonation wave segment DO and segments of characteristics CO of the first, and DC of the second set. It is easy to verify, when considering the validity of this solution in the region of positive values of τ beyond the characteristic CO, that conditions at the detonation wave will be fulfilled, if the wave equation for $\tau > 0$ is taken in the form

$$\lambda_D = 1 + d_3^{\circ}\tau^3 + \cdots, \qquad d_3^{\circ} = d\,\frac{V_1^{\circ 2} + 2d}{V_1^{\circ 2} - 2d}$$

With this, the detonation wave will be super-compressed everywhere, with the exception of point O. At point O, at which the Chapman – Jouguet mode is obtained, the detonation wave will have an inflection point. The wave form for the case of positive τ is shown on Fig. 2 by the dotted line OD_1.

Let us assume that behind the characteristic CO the derived flow is joined by another in which the velocity increase along segment CC_1 of characteristic DCC_1 differs from that of the analytical extension of the flow from the DCO region.

Let us determine the values of functions V, P, and R along the CO characteristic. For this purpose we shall, first of all find equations of the characteristic for the generalized case of flows defined by Formulas (19) to (22). Using expansions (14) and (19), and also Eqs. (20) and (21), we obtain from relationships (12a), after certain transformations, the following equations for the characteristics of the first and second sets:

$$\frac{d\lambda}{d\tau} + \lambda = 1 + 2\lambda_2\tau + \frac{1-q}{2}V_{10}\sqrt{1 + \lambda_2\tau^2 + \cdots - \lambda} + \cdots$$

$$\frac{d\lambda}{d\tau} + \lambda = \frac{1-\gamma-2q}{\gamma+1} + 2\left(\frac{1-q}{\gamma+1}V_{01} - \lambda_2\right)\tau + \qquad (25)$$

$$+ \frac{(3-\gamma)(1-q)}{2(\gamma+1)}V_{10}\sqrt{1 + \lambda_2\tau^2 + \cdots - \lambda} + \cdots$$

Looking for the expression of the first set characteristic passing through point $\tau = 0$, $\lambda = 1$ in the form of

$$\lambda = 1 + c_1\tau + c_2\tau^2 + \ldots$$

we obtain

$$c_1 = 0, \qquad (c_2 - \lambda_2)\tau = {}^1/_4 (1-q) V_{10} \sqrt{\lambda_2 - c_2}\,|\tau|$$

It follows from this that

$$\lambda = 1 + [\lambda_2 - {}^1/_{16} (1-q)^2 V_{10}^2]\tau^2 + \ldots \tag{26}$$

The right-hand segment of curve (26) is to be taken for $V_{10} > 0$, with $\tau < 0$, and the left-hand segment of this curve for $V_{10} < 0$ with $\tau > 0$.

The characteristic Eq. (26) for flows behind a detonation wave defined by Formulas (24) becomes

$$\lambda = 1 - {}^1/_{16} (1-q)^2 V_1^{\circ 2} \tau^2 + \ldots$$

while for velocity V along the characteristic we obtain

$$V = 1 - {}^1/_4 (1-q) V_1^{\circ 2} \tau + \ldots \tag{27}$$

Velocity distribution (as well as that of pressure and density) and characteristic CO will thus be independent of parameter d_3, and will be the same as those in a self-similar flow behind a Chapman – Jouguet wave. This makes it possible to extend the flow defined for a certain $d_3 > 0$ by Formulas (24) beyond region DCO, and to join it along characteristic CO to a flow defined by the same Formulas (24), but with d_3 of a different value. With this the derivative of the detonation curvature will suffer a discontinuity at point O. Weak discontinuities will also occur in the flow region, expanding from point O along the characteristic of the second set, and along the trajectory. Such discontinuities do not, however, appear if only the first two terms of expansions (12) are taken into account. In particular, if an extension of the flow beyond the characteristic with $d_3 = 0$, i.e. of a self-similar compression flow is considered, then the detonation wave will remain a Chapman – Jouguet wave even beyond point O.

Solutions (24) are not applicable to the case of a flow extending beyond region DCO, if the velocity increase along segment CC_1 of the second set characteristic is lower than that of a self-similar compression flow, as in these solutions $V_{10} > V_1^{\circ}$ for all values of d_3. In this case we have behind the Chapman – Jouguet wave OJ a region bounded by either the compression shock, or the first set characteristic emanating from point O, in which a self-similar rarefaction wave appears.

Let the equation of the discontinuity line coming out of point O be

$$\lambda_s = 1 - {}^1/_{16} (1-q)^2 V_1^{\circ 2} A^2 \tau^2 + \ldots \tag{28}$$

From what was said before about the characteristic of the first set in a rarefaction flow, it follows that $A^2 \leqslant 1$.

Along the discontinuity line the following conservation laws must be satisfied

$$\rho_1 (v_1 - c) = \rho_2 (v_2 - c), \qquad \rho_1 (v_1 - c)^2 + p_1 = \rho_2 (v_2 - c)^2 + p_2$$

$$\frac{1}{2}(v_1 - c)^2 + \frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} = \frac{1}{2}(v_2 - c)^2 + \frac{\gamma}{\gamma - 1}\frac{p_2}{\rho_2}$$

With the variables used here these relationships, when solved for parameters behind the discontinuity (superscript $+$), become

$$V^+ = \frac{\gamma - 1}{\gamma + 1} V + \frac{2}{1-q}\frac{c}{c_J} + \frac{2(\gamma + q)^2}{(\gamma + 1)(1-q)^2}\frac{P}{R[V - (\gamma + 1)c/(1-q)c_J]}$$

$$P^+ = -\frac{\gamma - 1}{\gamma + 1} P + \frac{2\gamma(1-q)^2}{(\gamma + 1)(\gamma + q)^2} R\left(V - \frac{\gamma + 1}{1-q}\frac{c}{c_J}\right)^2 \tag{29}$$

$$R^+ = R\left(V - \frac{\gamma + 1}{1-q}\frac{c}{c_J}\right)^2 \left[\frac{\gamma - 1}{\gamma + 1}\left(V - \frac{\gamma + 1}{1-q}\frac{c}{c_J}\right)^2 + \frac{2(\gamma + q)^2}{(\gamma + 1)(1-q)^2}\frac{P}{R}\right]^{-1}$$

Parameters V, R and P correspond to the already known self-similar rarefaction flow in front of the discontinuity, and $c/c_J = \lambda_s' + \lambda_s$.

A substitution of expressions for V, P, R, and λ_s into Eqs. (29) yields

$$V^+ = 1 + \alpha\tau + \cdots, \quad P^+ = 1 + \gamma\frac{1-q}{\gamma+q}\alpha\tau + \cdots, \quad R^+ = 1 + \frac{1-q}{\gamma+q}\alpha\tau + \cdots$$

$$\alpha = {}^1/_4 (1-q) V_1^{\circ 2} A (1-2A) \tag{30}$$

Looking again for a solution behind the discontinuity line in the form of (19), we take into consideration conditions (20) to (22), stipulate the fulfillment of conditions (30) for $\lambda = \lambda_*$, and find the same Expressions (24) for the definition of all parameters in terms of V_{10}^+, while the interdependence of parameters V_{10}^+ and A, defining the discontinuity line curvature, will, of course, differ from that of the last relationship of (24), having the form (superscript $^+$ has been omitted)

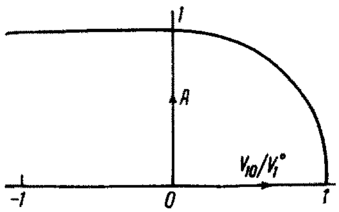

Fig. 4

$$V_{10}^2 - V_1^{\circ 2} + V_{10}\sqrt{V_{10}^2 - (1-A^2)V_1^{\circ 2}} = V_1^{\circ 2}A(1-2A)$$

This dependence is shown diagramatically on Fig. 4. For $A = 0$ the flow behind a Chapman – Jouguet wave is a self-similar compression flow with $V_{10} = V_1^\circ$. With increasing A parameter V_{10} decreases, becoming zero for $A = 1$. The compression shock bounding the self-similar rarefaction flow behind the Chapman – Jouguet wave degenerates with this into a characteristic. With a further decrease of V_{10} the equality $A = 1$ remains unchanged, but the flow joining the characteristic changes, becoming an analytical extension of the self-similar rarefaction flow in front of the characteristic for the particular case of $V_{10} = -V_1^\circ$. With a still further decrease of V_{10} the characteristic is joined by a flow with a stronger rarefaction than that of a self-similar wave. For positive values of parameter V_{10}, i.e. when discontinuity OS is a compression shock, the derived solutions, dependent on parameter A (i.e. on the curvature of the discontinuity line OS), may be joined in a continuous manner along characteristic CO to the flow behind the detonation wave defined by Eqs. (24).

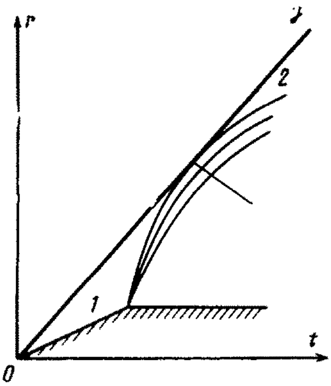

Fig. 5

It can be easily verified that along characteristic CO the velocity in the flow behind the discontinuity is expressed by

$$V = 1 + V_{01}^+\tau + V_{10}^+\sqrt{[\lambda_2 + {}^1/_{16}(1-q)^2 V_1^{\circ 2}]\tau^2} + \cdots =$$

$$= 1 + {}^1/_4(1-q)(V_{10}^2 - V_1^{\circ 2})\tau + {}^1/_4(1-q)V_{10}^+|V_{10}^+\tau| + \cdots \tag{31}$$

In accordance with this expression the value of V for $V_{10}^+ > 0$ is defined by Formula

$$V = 1 - {}^1/_4(1-q)V_1^{\circ 2}\tau + \ldots$$

and coincides with that defined by Formula (27) for flows behind a detonation wave. This is evidently true for the pressure and density parameters. We note that the derivatives of V, P, and R with respect to λ remain continuous when crossing characteristic CO.

Values of these derivatives, as shown on the example of the derivative of V

$$\left.\frac{\partial V}{\partial \lambda}\right|_c = \frac{2}{(1-q)\tau} + \cdots$$

are independent of parameters A and d_3.

Characteristic (28) and line $\lambda = \lambda_0(\tau)$ coincide when $V_{10}^+ = 0$. The solution behind the characteristic may be sought in the form given by (13), and with the aid of Formulas (24) for V_{01} and λ_2 we find that

$$V = 1 - \frac{1-q}{4} V_1^{\circ 2}\tau + V_{20}^* \left[\lambda - 1 + \left(\frac{1-q}{4}\right)^2 V_1^{\circ 2}\tau^2\right] + \cdots$$

Here V_{20}^* is arbitrary. For any V_{20}^* this solution joins continuously the solution in region DCO. The flow between characteristics CO and OS does not have any singularities for $\tau = 0$, $\lambda = 1$.

For $V_{10}^+ < 0$ solutions behind the characteristic OS are given by expansions (14), and have a singularity at point $\tau = 0$, $\lambda = 1$.

It follows from Eq. (31) for the velocity along characteristic CO that for $V_{10}^+ < 0$ the solution behind characteristic OS cannot be continuously joined to the solution behind a detonation wave along characteristic CO. It is easy to see that a continuous joining of these two solutions along the second set characteristic originating at the Chapman – Jouguet point is not possible either. In accordance with the second Formula of (25) the equation of such a characteristic may be obtained in the following form

$$\lambda = 1 - 2\frac{\gamma+q}{\gamma+1}\tau + \frac{(3-\gamma)(1-q)}{3(\gamma+1)} V_{10} \left(2\frac{\gamma+q}{\gamma+1}\right)^{1/2} \tau^{3/2} + \cdots$$

Because of the assumption that $V_{10}^+ < 0$, while in the flow behind a detonation wave $V_{10} > 0$, this characteristic is different for each of the flows considered. We note that a continuous joining of solutions along the second set characteristic is not possible when $V_{10}^+ > 0$.

The derived solution defines in particular the transition of a self-similar compression flow *1* (Fig. 5) with a Chapman – Jouguet wave OJ, originated by the piston expansion at a corresponding constant velocity, into a rarefaction flow *2* with a Chapman – Jouguet wave which develops after the piston has come to rest.

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Translated by J.J.D.